The bridge survival game in *Squid Game*

E. Adrian Henle[1], Nick Gantzler[2], François-Xavier Coudert[3], and Cory M. Simon[1]

[1]School of Chemical, Biological, and Environmental Engineering. Oregon State University. Corvallis, OR. USA.
[2]Department of Physics. Oregon State University. Corvallis, OR. USA.
[3]Chimie ParisTech, PSL University, CNRS, Institut de Recherche de Chimie Paris, Paris, France

November 15, 2021

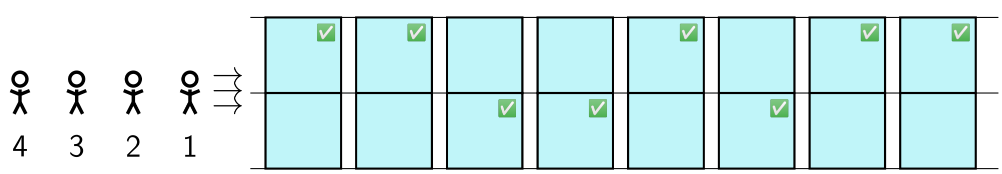
Abstract

In episode seven ("VIPs") of the television series *Squid Game*, the characters play a game of chance where they must traverse a bridge composed of glass panels—some sufficiently strong to support the weight of a player, some not. Inspired by *Squid Game*, we frame and analyze a stochastic process, "The Bridge Survival Game".

The Bridge Survival Game

In the Bridge Survival Game, N players line up to sequentially attempt to cross a bridge.

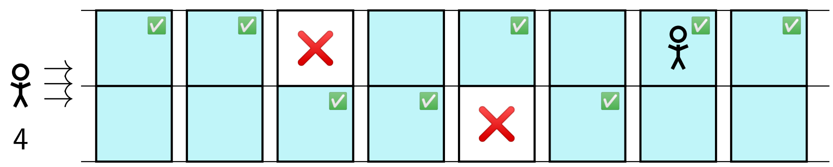
The bridge is comprised of a $2 \times L$ grid of glass panels. Half of the glass panels are strong (tempered) and can support the weight of a player; the other half are weak (untempered) and shatter under the weight of a player. The strong glass panels are randomly distributed on the bridge, under the constraint that each column contains exactly one strong glass panel. Owing to spacing between the columns, each player must traverse the bridge column-by-column, in L hops.



An initial condition for a Bridge Survival Game with $N = 4$ players attempting to traverse a bridge with $L = 8$ columns of glass panels. Strong glass panels are marked with ✅ (unseen to the players).

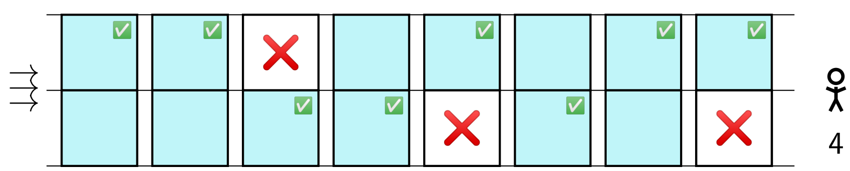
To the players, the glass panels are visually indistinguishable. Therefore, for each advance to an unvisited (by any player) column, the active player (the player at the front, currently attempting bridge traversal) chooses a glass panel at random on which to hop. If the player hops onto the strong glass panel, he/she proceeds to hop onto the next column. On the other hand, if the player hops onto the weak glass panel, the glass shatters, he/she is eliminated, and the player behind then becomes the active player to attempt traversal of the bridge.

The players behind observe the outcomes of the hops of players in front of them. Owing to perfect memory and survival instincts, if a player in front has successfully traversed a column by hopping onto a strong glass panel, the remaining players will hop onto the same sturdy panel in their [attempted] traversal of the bridge.



The active player, player three, is traversing the bridge and, by observing the outcomes of the hops of the (eliminated) two players in front of him/her, has taken only two *risky* hops to arrive at his/her current position.

The game proceeds until (i) all players are eliminated or (ii) all columns of the bridge have been traversed and a subset of the players safely cross the bridge.



Continuing with the scenario above, in this outcome (end game), only player four successfully traversed the bridge, without taking any *risky* hops.

❓ What is the probability that player $i \in \{1, 2, \ldots, N\}$ (where player 1 is the first to attempt bridge traversal) survives by successfully crossing the length-L bridge?

Analysis

💡 Regardless of its outcome, each hop onto an unvisited (by any player) column by an active player constitutes an observation uncovering which panel in that column is composed of strong glass. From the perspective of the group of N players (observers), this observation incurs a cost (elimination of a player) only if the active player chooses a weak glass panel.

Probability that exactly n players are eliminated

Let E_n be the event that *exactly* $n \in \{0, 1, 2, \ldots, N\}$ players are eliminated in the Bridge Survival Game. We wish to find the probability of this event, $P(E_n)$.

Case $N > L$

Suppose the number of players is greater than the bridge length ($N > L$).

At most, L players can be eliminated, by all of the first L players choosing a weak glass panel in their first hop onto an unvisited column. This outcome would reveal the path for safe traversal over the bridge for the remaining $N - L$ players behind them. Therefore, $P(E_n) = 0$ for $n > L$.

Given $n \leq L$ players were eliminated, there are $\binom{L}{n}$ ways to distribute the n broken glass panels among the L columns of the bridge. Each *particular* distribution of broken panels occurs with probability $\left(\frac{1}{2}\right)^L$ because it corresponds to a distinct sequence of outcomes of L hops to previously unvisited (by all players) columns, each of which is an independent event with probability $\frac{1}{2}$. Note, these L risky hops ("observations") were taken by (i) n players, if a panel was broken in column L (the last), or (ii) $n+1$ players, if a panel in column L was not broken. The outcomes corresponding to the set of distributions of n broken panels on the bridge are mutually exclusive. Therefore:

$$P(E_n) = \begin{cases} \binom{L}{n}\left(\frac{1}{2}\right)^L & 0 \leq n \leq L \\ 0 & L < n \leq N. \end{cases} \tag{1}$$

Another way to arrive at eqn. 1: of 2^L equally likely outcomes, $\binom{L}{n}$ of them result in n eliminated players.

Two sanity checks on eqn. 1: (1) The probability that zero players are eliminated is $P(E_0) = \left(\frac{1}{2}\right)^L$, since then the first player must choose the strong glass panel in each column for each of their L hops to cross the bridge; by similar reasoning, $P(E_L) = \left(\frac{1}{2}\right)^L$. (2) Since the events $\{E_0, E_1, ..., E_N\}$ are mutually exclusive and their union comprises the sample space, we must have $\sum_{n=0}^{N} P(E_n) = 1$, which holds since $\sum_{n=0}^{L} \binom{L}{n} = 2^L$ via the binomial theorem.

Case $N \leq L$

Suppose the number of players is less than or equal to the bridge length ($N \leq L$). Unlike the case above, no player is certain to survive.

Given $n < N$ players were eliminated (i.e., at least one player survived), the argument above and eqn. 1 for $P(E_n)$ holds, since all L columns of the bridge must have been visited. However, the event E_N (all players eliminated) is special because, then, all L columns of the bridge were not necessarily visited (if $N < L$). For the event E_N, we consider the set of events that player N was eliminated at column $c \in \{N, N+1, ..., L\}$ of the bridge. The union of these mutually exclusive events comprise the sample space for player N given they did not survive. For each column c at which player N was eliminated, there are $\binom{c-1}{N-1}$ ways to distribute the broken glass panels of the remaining $N-1$ players onto the previous $c-1$ columns on the bridge; and, each distribution corresponds to a distinct sequence of particular outcomes of c independent risky hops to unvisited (by any player) columns. Therefore,

$$P(E_n) = \begin{cases} \binom{L}{n}\left(\frac{1}{2}\right)^L & 0 \leq n < N \\ \sum_{c=N}^{L} \binom{c-1}{N-1}\left(\frac{1}{2}\right)^c & n = N. \end{cases} \tag{2}$$

Two sanity checks on eqn. 2: (1) It reduces to eqn. 1 when $N = L$. (2) Again, $\sum_{n=0}^{N} P(E_n) = 1$, as we show in the Appendix.

Probability that player i survives

Let S_i be the event that player i survives. This event is the union of the events that $n \in \{0, 1, ..., i-1\}$ players are eliminated:

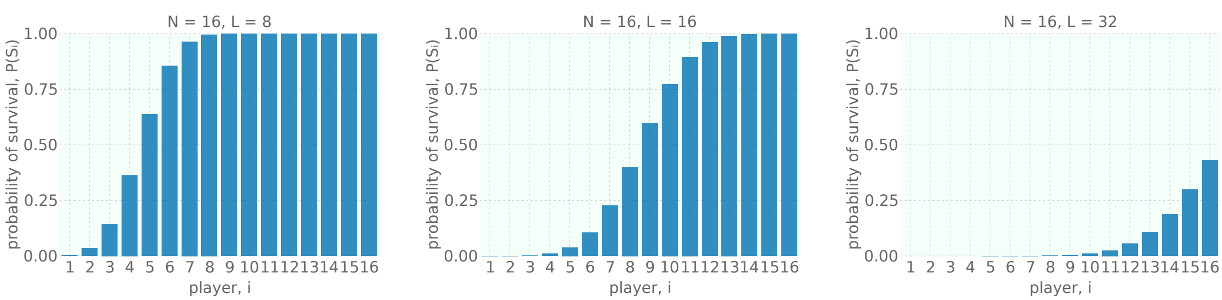
$$S_i = \cup_{n=0}^{i-1} E_n. \tag{3}$$

Since the events $\{E_0, E_1, ..., E_N\}$ are mutually exclusive:

$$P(S_i) = \sum_{n=0}^{i-1} P(E_n) = \begin{cases} \sum_{n=0}^{i-1} \binom{L}{n}\left(\frac{1}{2}\right)^L & 1 \leq i \leq \min(L, N) \\ 1 & i > L, N > L. \end{cases} \tag{4}$$

Eqn. 4 holds for both $N \geq L$ and $N < L$, and $P(E_N)$ is not involved.

Fig. 1 shows $P(S_i)$ under three different scenarios: (a) $N > L$, (b) $N = L$, and (c) $N < L$. The players at the front of the line to cross the bridge have the lowest probability of survival.



(a) case $N > L$ (b) case $N = L$ (c) case $N < L$

Figure 1: We visualize $P(S_i)$, the probability that player i of $N = 16$ total players, survives the bridge game for a length $L \in \{8, 16, 32\}$ bridge, according to eqn. 4.

Expected number of players that survive

Let the random variable $\theta \in \{0, 1, ..., N\}$ be the number of players that successfully traverse the bridge. The expected value of θ is:

$$\mathbb{E}[\theta] = \sum_{n=0}^{N} (N - n) P(E_n), \tag{5}$$

with $P(E_n)$ given in eqn. 2 since (i) the [union of the] mutually exclusive events $\{E_0, E_1, ..., E_N\}$ comprise the sample space and (ii) exactly n players eliminated implies $N - n$ players survived. Fig. 2 shows $\mathbb{E}[\theta]$ as a function of the bridge length, L, for $N = 16$ total players. As the bridge lengthens, fewer players are expected to survive.

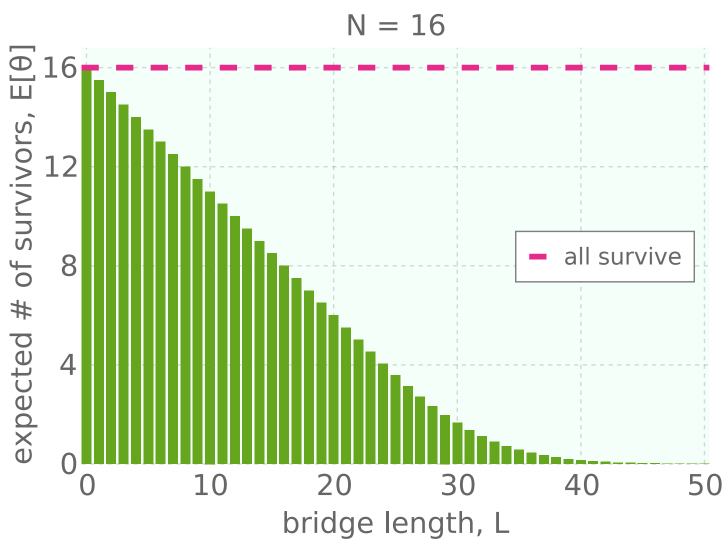


Figure 2: We visualize the expected number of players that survive, $\mathbb{E}[\theta]$ in eqn. 5, as a function of bridge length L, for $N = 16$ total players.

Extensions

Extensions to the Bridge Survival Game include:

- incorporating an ability for players to stochastically discern between strong vs. weak glass panels
- accounting for imperfect memory of the players observing the outcomes of the players in front of them (i.e., faulty memory of which glass panel of a previously- and successfully-traversed column is the strong one)
- a bridge with multiple rows and/or varying distributions of weak vs. strong glass panels.

Appendix

We wish to prove:

$$\sum_{n=0}^{N} P(E_n) = 1 \tag{6}$$

for $N < L$, with $P(E_n)$ in eqn. 2. We expect this to hold because the events $\{E_0, E_1, ..., E_N\}$ are disjoint and their union comprises the sample space. Using eqn. 2, we wish to show:

$$\sum_{n=0}^{N-1} \binom{L}{n} \left(\frac{1}{2}\right)^L + \sum_{c=N}^{L} \binom{c-1}{N-1} \left(\frac{1}{2}\right)^c = 1. \tag{7}$$

Via the binomial theorem:

$$\sum_{n=0}^{L} \binom{L}{n} = 2^L. \tag{8}$$

Splitting the sum gives:

$$\sum_{n=0}^{N-1} \binom{L}{n} = 2^L - \sum_{n=N}^{L} \binom{L}{n} \tag{9}$$

Substituting this expression into eqn. 7, we equivalently want to show:

$$2^L \sum_{c=N}^{L} \binom{c-1}{N-1} \left(\frac{1}{2}\right)^c = \sum_{n=N}^{L} \binom{L}{n}. \tag{10}$$

We proceed by manipulating the term on the right to transform it into the term on the left.

We use Pascal's identity:

$$\binom{L}{n} = \binom{L-1}{n} + \binom{L-1}{n-1} \tag{11}$$

to write:

$$\sum_{n=N}^{L} \binom{L}{n} = \sum_{n=N}^{L-1} \binom{L}{n} + 1 \tag{12}$$

$$= \sum_{n=N}^{L-1} \left[\binom{L-1}{n} + \binom{L-1}{n-1} \right] + 1 \tag{13}$$

$$= \sum_{n=N}^{L-1} \binom{L-1}{n} + \binom{L-1}{N-1} + \sum_{n=N+1}^{L} \binom{L-1}{n-1}. \tag{14}$$

Reindexing the second sum gives:

$$\sum_{n=N}^{L} \binom{L}{n} = \binom{L-1}{N-1} + 2 \sum_{n=N}^{L-1} \binom{L-1}{n}. \tag{15}$$

By defining $f : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$:

$$f(L, N) := \sum_{n=N}^{L} \binom{L}{n}, \tag{16}$$

eqn. 15 provides a recurrence relation:

$$f(L, N) = \binom{L-1}{N-1} + 2f(L-1, N). \tag{17}$$

Applying the recurrence relation repeatedly gives:

$$\sum_{n=N}^{L} \binom{L}{n} = \sum_{n=1}^{L-N+1} 2^{n-1} \binom{L-n}{N-1}. \tag{18}$$

Reindexing the sum via $L - n =: c - 1$ gives:

$$\sum_{n=N}^{L} \binom{L}{n} = 2^L \sum_{c=N}^{L} \binom{c-1}{N-1} \left(\frac{1}{2}\right)^c \tag{19}$$

as we desired in eqn. 10.